

QUATERNION GEOMETRIES ON THE TWISTOR SPACE OF THE SIX-SPHERE

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ABSTRACT. We explicitly describe all $SO(7)$ -invariant almost quaternion-Hermitian structures on the twistor space of the six sphere and determine the types of their intrinsic torsion.

CONTENTS

1. Introduction	1
2. Invariant forms	1
3. Intrinsic torsion	3
4. Explicit structures	4
References	7

1. INTRODUCTION

Recently Moroianu, Pilca and Semmelmann [5] found that the twistor space $M = SO(7)/U(3)$ of the six sphere S^6 admits a homogeneous almost quaternion-Hermitian structure. This arose as part of their striking result that M is the only such homogeneous space with non-zero Euler characteristic that is neither quaternionic Kähler (the quaternionic symmetric spaces of Wolf [9]) nor $S^2 \times S^2$.

In this paper we show that there is exactly a one-dimensional family of invariant almost quaternion-Hermitian structures on M , with fixed volume, and determine the types of their intrinsic torsion. We will see that the family contains inequivalent structures, and includes the symmetric Kähler metric of the quadric $\widetilde{\text{Gr}}_2(\mathbb{R}^6) = SO(8)/SO(2)SO(6)$. Each member of the family will be shown to have almost quaternion-Hermitian type $\Lambda_0^3 E(S^3 H + H)$ with the first component non-zero, confirming that they are not quaternionic Kähler; one member of the family has pure type $\Lambda_0^3 E S^3 H$, and this is the first known example of such a geometry. However, the structure singled out by this almost quaternionic-Hermitian intrinsic torsion is not the Kähler metric of the quadric nor the squashed Einstein metric in the canonical variation.

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2. INVARIANT FORMS

The subgroup $U(3)$ of $SO(7)$ arises from a choice of identification of \mathbb{R}^7 as $\mathbb{R} \oplus \mathbb{C}^3$. Regarding $U(3)$ as $U(1)SU(3)$, we may write $\mathbb{C}^3 = \mathbb{R}^6 = \llbracket L\lambda^{1,0} \rrbracket$,

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meaning that $\mathbb{R}^6 \otimes \mathbb{C} = L\lambda^{1,0} + \overline{L\lambda^{1,0}} \cong L\lambda^{1,0} + L^{-1}\lambda^{0,1}$, where $L = \mathbb{C}$ and $\lambda^{1,0} = \mathbb{C}^3$ as the standard representations of $U(1)$ and $SU(3)$, respectively. We thus have $U(3) \leq SO(6) \leq SO(7)$, so $M = SO(7)/U(3)$ fibres over $S^6 = SO(7)/SO(6)$ with fibre $SO(6)/U(3)$, the almost complex structures on $T_x S^6$. Thus M is the (Riemannian) twistor space of S^6 .

Since $\lambda^{3,0} = \Lambda^3 \lambda^{1,0} = \mathbb{C}$ is trivial, we have $\lambda^{2,0} \cong \lambda^{0,1}$ as $SU(3)$ -modules. The Lie algebra of $SO(7)$ now decomposes as

$$\begin{aligned} \mathfrak{so}(7) &= \Lambda^2 \mathbb{R}^7 = \Lambda^2(\mathbb{R} + \llbracket L\lambda^{1,0} \rrbracket) = \llbracket L\lambda^{1,0} \rrbracket + \llbracket L^2\lambda^{2,0} \rrbracket + [\lambda^{1,1}] \\ &\cong \llbracket L\lambda^{1,0} \rrbracket + \llbracket L^2\lambda^{0,1} \rrbracket + \mathfrak{u}(1) + \mathfrak{su}(3). \end{aligned}$$

Here $[\lambda^{1,1}]$ is the real module whose complexification is $\lambda^{1,1} = \lambda^{1,0} \otimes \lambda^{0,1}$; it splits in to two irreducible modules $[\lambda_0^{1,1}] \cong \mathfrak{su}(3)$ and $\mathbb{R} = \mathfrak{u}(1)$.

We thus have that the complexified tangent space of $M = SO(7)/U(3)$ is the bundle associated to

$$\begin{aligned} T \otimes \mathbb{C} &= (\llbracket L\lambda^{1,0} \rrbracket + \llbracket L^2\lambda^{0,1} \rrbracket) \otimes \mathbb{C} \\ &= L\lambda^{1,0} + L^{-1}\lambda^{0,1} + L^2\lambda^{0,1} + L^{-2}\lambda^{1,0} \\ &= (L^{1/2}\lambda^{0,1} + L^{-1/2}\lambda^{1,0})(L^{3/2} + L^{-3/2}). \end{aligned} \tag{2.1}$$

This allows us to write $T \otimes \mathbb{C} = EH$, where $E = L^{1/2}\lambda^{0,1} + L^{-1/2}\lambda^{1,0}$ and $H = L^{3/2} + L^{-3/2}$ are representations of $U(1)_2 \times SU(3)$ as a subgroup of $U(1)_L SU(3) \times U(1)_R \leq Sp(3) \times Sp(1)$. Here $U(1)_2$ is a double cover of $U(1)$ and is included in $U(1)_L \times U(1)_R$ via the map $e^{i\theta} \mapsto (e^{-i\theta}, e^{3i\theta})$. In this way, we see that $M = SO(7)/U(3)$ carries an invariant $Sp(3)Sp(1)$ -structure, where $Sp(3)Sp(1) = (Sp(3) \times Sp(1))/\{\pm(1, 1)\}$. This is the G -structure description of an almost quaternion-Hermitian structure.

Geometrically an almost quaternion-Hermitian structure is specified by a Riemannian metric g and a three-dimensional subbundle \mathcal{G} of $\text{End}(TM)$ which locally has a basis I, J, K satisfying the quaternion identities

$$I^2 = -1 = J^2, \quad IJ = K = -JI$$

and the compatibility conditions

$$g(I\cdot, I\cdot) = g(\cdot, \cdot) = g(J\cdot, J\cdot).$$

There are then local two-forms

$$\begin{aligned} \omega_I(X, Y) &= g(X, IY), \quad \omega_J(X, Y) = g(X, JY), \\ \omega_K(X, Y) &= g(X, KY) \end{aligned}$$

and with the local form $\omega_c = \omega_J + i\omega_K$ of type $(2, 0)$ with respect to I . Since they are non-degenerate, the local forms $\omega_I, \omega_J, \omega_K$ are sufficient to determine the local almost complex structures I, J and K and the metric g .

Equation (2.1), show us that T has two inequivalent irreducible summands $\llbracket L\lambda^{1,0} \rrbracket$ and $\llbracket L^2\lambda^{0,1} \rrbracket$ and so there are two invariant forms ω_0 and $\tilde{\omega}_0$ spanning $\Omega^2(M)^{SO(7)}$. However, we have that

$$\begin{aligned} \Lambda^2 T &= \Lambda^2 \llbracket L\lambda^{1,0} \rrbracket + \Lambda^2 \llbracket L^2\lambda^{0,1} \rrbracket + \llbracket L\lambda^{1,0} \rrbracket \wedge \llbracket L^2\lambda^{0,1} \rrbracket \\ &= (\mathbb{R}\omega_0 + [\lambda_0^{1,1}] + \llbracket L^2\lambda^{0,1} \rrbracket) + (\mathbb{R}\tilde{\omega}_0 + [\lambda_0^{1,1}] + \llbracket L^4\lambda^{1,0} \rrbracket) \\ &\quad + (\llbracket L^3 \rrbracket + \llbracket L^3 \rrbracket [\lambda_0^{1,1}] + \llbracket L\lambda^{1,0} \rrbracket + \llbracket L\sigma^{0,2} \rrbracket), \end{aligned} \tag{2.2}$$

where $\sigma^{0,2} = S^2\lambda^{0,1}$. There is thus an additional 2-dimensional subspace $\llbracket L^3 \rrbracket$ preserved by the $SU(3)$ -action. This space is spanned by local $SU(3)$ -invariant forms ω_J and ω_K , that are mixed under the $U(1)$ -action, so that $\omega_c = \omega_J + i\omega_K$ is a basis element of L^3 . We may now consider the triple of forms

$$\omega_I = \lambda\omega_0 + \mu\tilde{\omega}_0, \quad \omega_J \quad \text{and} \quad \omega_K \quad (2.3)$$

which will be seen to result in an almost quaternion-Hermitian structure when

$$20\lambda^3\mu^3(\omega_0)^3(\tilde{\omega}_0)^3 = (\omega_J)^6. \quad (2.4)$$

This equation is necessary, as each two form in the triple must define the same volume element.

We note that for an almost quaternion-Hermitian structure the four-form $\Omega = \omega_I^2 + \omega_J^2 + \omega_K^2$ is globally defined. For an invariant structure, this form must lie in $\Omega^4(M)^{SO(7)}$ which in our particular case is four-dimensional. Indeed the complete decomposition of $\Lambda^4 T$ into irreducible $U(3)$ -modules is

$$\begin{aligned} \Lambda^4 T = & \llbracket L^6 \rrbracket + 2\llbracket L^3 \rrbracket + 4\mathbb{R} \\ & + \llbracket L^7\lambda^{1,0} \rrbracket + 3\llbracket L^4\lambda^{1,0} \rrbracket + 5\llbracket L\lambda^{1,0} \rrbracket + 4\llbracket L^2\lambda^{0,1} \rrbracket + 2\llbracket L^5\lambda^{0,1} \rrbracket \\ & + 2\llbracket L^2\sigma^{2,0} \rrbracket + 2\llbracket L\sigma^{0,2} \rrbracket + \llbracket L^4\sigma^{0,2} \rrbracket \\ & + \llbracket L^3\sigma^{3,0} \rrbracket + \llbracket \sigma^{3,0} \rrbracket + \llbracket L^3\sigma^{0,3} \rrbracket \\ & + \llbracket L^6\lambda_0^{1,1} \rrbracket + 4\llbracket L^3\lambda_0^{1,1} \rrbracket + 6\llbracket \lambda_0^{1,1} \rrbracket \\ & + \llbracket L^4\sigma_0^{2,1} \rrbracket + 2\llbracket L^2\sigma_0^{2,1} \rrbracket + \llbracket L^2\sigma_0^{1,2} \rrbracket + \llbracket \sigma_0^{2,2} \rrbracket. \end{aligned}$$

Now the four-forms ω_0^2 , $\tilde{\omega}_0^2$, $\omega_0 \wedge \tilde{\omega}_0$ and $\omega_J^2 + \omega_K^2$ are invariant and linearly independent, so they provide a basis for $\Omega^4(M)^{SO(7)}$. It follows, Lemma 4.1 below, that any invariant almost hyperHermitian structure on M is described via the forms of (2.3).

3. INTRINSIC TORSION

Given an invariant almost Hermitian structure on M , there is a unique $Sp(3)Sp(1)$ -connection ∇ characterised by the condition that the pointwise norm of its torsion is the least possible. More precisely, ∇ is related to the Levi-Civita connection by

$$\nabla = \nabla^{\text{LC}} + \xi,$$

where ξ is the intrinsic torsion given [4] by

$$\xi_X Y = -\frac{1}{4} \sum_{A=I,J,K} A(\nabla_X^{\text{LC}} A)Y + \frac{1}{2} \sum_{A=I,J,K} \lambda_A(X)AY,$$

with

$$6\lambda_I(X) = g(\nabla_X^{\text{LC}} \omega_J, \omega_K),$$

etc. The tensor ξ takes values in

$$\mathcal{Q} = T^* \otimes (\mathfrak{sp}(3) + \mathfrak{sp}(1))^\perp \subset T^* \otimes \Lambda^2 T^*$$

where $\mathfrak{sp}(3) = [S^2 E]$ and $\mathfrak{sp}(1) = [S^2 H]$ are the Lie algebras of $Sp(3)$ and $Sp(1)$. Under the action of $Sp(3)Sp(1)$, the space $\mathcal{Q} \otimes \mathbb{C}$ decomposes as

$$\mathcal{Q} \otimes \mathbb{C} = (\Lambda_0^3 E + K + E)(S^3 H + H)$$

with $\Lambda_0^3 E$ and K irreducible $Sp(3)$ -modules satisfying $\Lambda^3 E = \Lambda_0^3 E + E$ and $E \otimes S^2 E = S^3 E + K + E$. The space \mathcal{Q} thus has six irreducible summands under $Sp(3)Sp(1)$.

For an invariant structure on $M = SO(7)/U(3)$, the intrinsic torsion lies in a $U(3)$ -invariant submodule of \mathcal{Q} . As $\mathfrak{sp}(3) = [S^2(L^{1/2}\lambda^{0,1})] = \llbracket L\sigma^{0,2} \rrbracket + [\lambda_0^{1,1}] + \mathbb{R}$ and $\mathfrak{sp}(1) = [S^2(L^{3/2})] = \llbracket L^3 \rrbracket + \mathbb{R}$, equation (2.2), implies that

$$(\mathfrak{sp}(3) + \mathfrak{sp}(1))^\perp \cong [\lambda_0^{1,1}] + \llbracket L^2\lambda^{0,1} \rrbracket + \llbracket L^4\lambda^{1,0} \rrbracket + \llbracket L^3 \rrbracket [\lambda_0^{1,1}] + \llbracket L\lambda^{1,0} \rrbracket.$$

Comparing with equation (2.1), we see that $(\mathfrak{sp}(3) + \mathfrak{sp}(1))^\perp$ contains a unique copy of each of the irreducible summands of T , so $\mathcal{Q}^{U(3)}$ is two dimensional. As $\Lambda^3(A+B) \cong \Lambda^3 A + \Lambda^2 A \otimes B + A \otimes \Lambda^2 B + \Lambda^3 B$, we find that

$$\Lambda_0^3 E = (L^{3/2} + L^{-3/2}) + (L^{1/2}\sigma^{2,0} + L^{-1/2}\sigma^{0,2}).$$

The first summand is a copy of H and is also a submodule of $S^3 H = L^{9/2} + L^{3/2} + L^{-3/2} + L^{-9/2}$. This shows that $[\Lambda_0^3 E S^3 H]^{U(3)}$ and $[\Lambda_0^3 E H]^{U(3)}$ are each one-dimensional, and so we have

$$\xi \in \mathcal{Q}^{U(3)} \subset [\Lambda_0^3 E S^3 H] + [\Lambda_0^3 E H]. \quad (3.1)$$

4. EXPLICIT STRUCTURES

We now wish to determine the components of ξ in each of the summands of (3.1). An invariant almost Hermitian structure on M , may be described by two-forms as in (2.3). As ω_J and ω_K are only invariant under $SU(3)$, they do not define global forms on M . However, we do get two such invariant forms on the total space of the circle bundle $N = SO(7)/SU(3) \rightarrow M = SO(7)/U(3)$.

Let $0, 1, 2, 3, 1', 2', 3'$ be an orthonormal basis for $\mathbb{R}^7 = \mathbb{R} + \mathbb{C}^3$, with $0 \in \mathbb{R}$ and $i1 = 1'$, etc. Writing 12 for $1 \wedge 2$, a standard basis for $\llbracket L\lambda^{1,0} \rrbracket \subset \mathfrak{so}(7)$ is given by

$$A = 01, \quad B = 02, \quad C = 03, \quad A' = 01', \quad B' = 02', \quad C' = 03'$$

and a corresponding basis for $\llbracket L^2\lambda^{0,1} \rrbracket$ is

$$\begin{aligned} P &= 23 - 2'3', & Q &= 31 - 3'1', & R &= 12 - 1'2', \\ P' &= 23' - 32', & Q' &= 31' - 13', & R' &= 12' - 21'. \end{aligned}$$

We put $E = 11' + 22' + 33'$, and note that this is a generator of the central $\mathfrak{u}(1)$ in $\mathfrak{u}(3)$. Then $\{E, A, \dots, R'\}$ is a basis for $\mathfrak{n} = T_{\text{Id } SU(3)} N$ and $\{A, \dots, R'\}$ is a basis for $\mathfrak{m} = T_{\text{Id } U(3)} M$. We use lower case letters to denote the corresponding dual bases of \mathfrak{n}^* and \mathfrak{m}^* . These give left-invariant one-forms on $SO(7)$, with $da(X, Y) = -a([X, Y])$ for $X, Y \in \mathfrak{so}(7)$, etc. We write

$$d_N a = (da)|_{\Lambda^2 \mathfrak{n}} \quad \text{and} \quad d_M a = (da)|_{\Lambda^2 \mathfrak{m}}$$

at $\text{Id} \in SO(7)$. For a left-invariant form $\alpha \in \Omega^k(SO(7))$, we have at $\text{Id} \in SO(7)$ that $d\alpha = d_N \alpha$ if α is right $SU(3)$ -invariant and $d\alpha = d_M \alpha$ if α is right $U(3)$ -invariant. For our choice of bases, we have

$$\begin{aligned} d_M a &= -b \wedge r + c \wedge q - b' \wedge r' + c' \wedge q', & d_M p &= -\frac{1}{2}(b \wedge c - b' \wedge c'), \\ d_M a' &= -b \wedge r' + c \wedge q' + b' \wedge r - c' \wedge q, & d_M p' &= -\frac{1}{2}(b \wedge c' + b' \wedge c) \end{aligned}$$

with the other derivatives obtained by applying the cyclic permutation $(a, a', p, p') \rightarrow (b, b', q, q') \rightarrow (c, c', r, r') \rightarrow (a, a', p, p')$. We use \mathfrak{S} to denote sums over this group of permutations.

The two-form ω_I of (2.3) is

$$\begin{aligned}\omega_I &= \lambda(a' \wedge a + b' \wedge b + c' \wedge c) + \mu(p' \wedge p + q' \wedge q + r' \wedge r) \\ &= \mathfrak{S}(\lambda a' \wedge a + \mu p' \wedge p).\end{aligned}$$

On N , we have the forms $\hat{\omega}_J$ and $\hat{\omega}_K$ given by

$$\hat{\omega}_J + i\hat{\omega}_K = \mathfrak{S}((p + ip') \wedge (a + ia')).$$

Choosing a local section s of $\pi: N \rightarrow M$ such that $s(\text{Id } U(3)) = \text{Id } SU(3)$ and $s^*e = 0$, we then obtain local two-forms

$$\omega_J = s^*\hat{\omega}_J, \quad \omega_K = s^*\hat{\omega}_K$$

completing the triple of (2.3). The corresponding metric on M is

$$g = \mathfrak{S}(\lambda(a^2 + a'^2) + \mu(p^2 + p'^2)) \quad (4.1)$$

and condition (2.4) is simply

$$\lambda\mu = 1. \quad (4.2)$$

These are the only invariant metrics on M with normalised volume form, since TM (2.1) has exactly two irreducible summands.

At $\text{Id } U(3)$, the almost complex structures satisfy

$$\begin{aligned}IA &= A', \quad IP = P', \quad J\frac{1}{\sqrt{\lambda}}A = \frac{1}{\sqrt{\mu}}P, \quad J\frac{1}{\sqrt{\lambda}}A' = -\frac{1}{\sqrt{\mu}}P', \\ K\frac{1}{\sqrt{\lambda}}A &= \frac{1}{\sqrt{\mu}}P', \quad K\frac{1}{\sqrt{\lambda}}A' = \frac{1}{\sqrt{\mu}}P.\end{aligned}$$

These act on forms via $Ia = -a(I\cdot)$, so with the normalisation condition (4.2), we have $Ja = \mu p$, $Jp = -\lambda a$, etc.

Lemma 4.1. *These describe all invariant almost quaternion-Hermitian structures on M with normalised volume form.*

Proof. We have noted above that (4.1) gives all the invariant metrics. Now the local almost complex structures, or equivalently their Hermitian two forms, associated to the almost quaternion Hermitian structure span a $U(3)$ -invariant subspace V of $\Lambda^2 T$ of dimension 3. Counting dimensions in the decomposition (2.2), shows that V is a subspace of $\mathbb{R}\omega_0 + \mathbb{R}\tilde{\omega}_0 + \llbracket L^3 \rrbracket$. In particular, $V \cap \llbracket L^3 \rrbracket$ is at least one-dimensional; $U(3)$ -invariance implies that $\llbracket L^3 \rrbracket \leq V$. As ω_J and ω_K are g -orthogonal of the same length for each normalised g in (4.1), we see that J and K are local almost complex structures belonging to the almost quaternion-Hermitian geometry. Finally, $I = JK$ is specified too. \square

Lemma 4.2. *For the choices of ω_I , ω_J and ω_K above normalised by (4.2) we have at the base point $\text{Id } U(3) \in M$ that*

$$\begin{aligned}Id\omega_I &= Id_M\omega_I = (\tfrac{1}{2}\mu - 2\lambda)\Phi, \\ Jd\omega_J &= 2\lambda\Phi - \tfrac{1}{2}\mu^3\Psi, \quad Kd\omega_K = 2\lambda\Phi + \tfrac{1}{2}\mu^3\Psi,\end{aligned}$$

where

$$\begin{aligned}\Phi &= \mathfrak{S}(a \wedge b \wedge r - a' \wedge b' \wedge r + a \wedge b' \wedge r' + a' \wedge b \wedge r'), \\ \Psi &= \mathfrak{S}(p \wedge q \wedge r - 3p \wedge q' \wedge r')\end{aligned}$$

and $Ad\omega_A(\cdot, \cdot, \cdot) = -d\omega_A(A\cdot, A\cdot, A\cdot)$, for $A = I, J, K$.

Proof. As ω_I is $U(3)$ -invariant we have $Id\omega_I = Id_M\omega_I$ which equals

$$(2\lambda - \frac{1}{2}\mu)I \mathfrak{S}(a \wedge b' \wedge r + a' \wedge b \wedge r - a \wedge b \wedge r' + a' \wedge b' \wedge r')$$

and gives the first claimed formula valid at any point of M .

For our choice of section s , we have at $\text{Id } U(3)$ that $Jd\omega_J = Js^*d_N\tilde{\omega}_J = Jd_M\tilde{\omega}_J$ which is

$$J \mathfrak{S}\left(-\frac{1}{2}a \wedge b \wedge c + \frac{3}{2}a \wedge b' \wedge c' + 2(a \wedge q \wedge r - a \wedge q' \wedge r' + a' \wedge q \wedge r' + a' \wedge q' \wedge r)\right).$$

Combined with the description of J , we thus get the claimed formula. The computation for $Kd\omega_K$ is similar. \square

To compute the intrinsic torsion we use the ‘minimal description’ of [4] which relies on computing the forms $\beta_I = Jd\omega_J + Kd\omega_K$, etc., and the contractions $\Lambda_A\beta_B$ of β_B with ω_A . For our structures, we have at the base point

$$\beta_I = 4\lambda\Phi, \quad \beta_J = \frac{1}{2}(\mu\Phi + \mu^3\Psi), \quad \beta_K = \frac{1}{2}(\mu\Phi - \mu^3\Psi)$$

and all contractions $\Lambda_A\beta_B = 0$. This confirms that the intrinsic torsion ξ has no components in $[E(S^3H + H)]$.

Theorem 4.3. *The component of ξ in $[\Lambda_0^3ES^3H]$ is always non-zero, so the almost quaternion-Hermitian is never quaternionic. The component of ξ in $[\Lambda_0^3EH]$ is zero if and only if $2\lambda = \mu$.*

Proof. Since we have shown in §3 that ξ has no component in $[K(S^3H + H)]$ and we saw above that each one form $\Lambda_A\beta_B$ is zero, at the base point, the results of [4] show that the $\Lambda_0^3ES^3H$ -component of ξ corresponds to

$$\psi^{(3)} := \frac{1}{12}(\beta_I + \beta_J + \beta_K) = \frac{1}{12}(4\lambda + \mu)\Phi$$

which is always non-zero under condition (4.2). The component in Λ_0^3EH is determined by

$$\psi_I^{(3)} := \frac{1}{8}(-\beta_I + 2(3 + \mathcal{L}_I)\psi^{(3)}),$$

where $\mathcal{L}_I = I_{(12)} + I_{(13)} + I_{(23)}$, with $I_{(12)}\alpha = \alpha(I\cdot, I\cdot, \cdot)$, etc. Now $\mathcal{L}_I\Phi = \Phi$, so

$$\psi_I^{(3)} = \frac{1}{12}(\mu - 2\lambda)\Phi$$

and the result follows. \square

Corollary 4.4. *The invariant almost quaternion-Hermitian structures on M are not quaternionic integrable, and their quaternionic twistor spaces are not complex.*

Proof. This follows directly from the following two facts [7]: (i) The underlying quaternionic structure is integrable if and only if the intrinsic torsion ξ has no S^3H component, i.e. it lies in $(\Lambda_0^3E + K + E)H$. (ii) The quaternionic twistor space is complex if and only if the underlying quaternionic structure is integrable. But we have shown the $\Lambda_0^3ES^3H$ -component of ξ is non-zero, so the result follows. \square

The almost Hermitian structure (g, ω_I) is easily seen to be integrable: $d_M(a + ia') = -(b - ib') \wedge (r + ir') + (c - ic') \wedge (q + iq') \in \Lambda_I^{1,1}$, $d_M(p + ip') = -\frac{1}{2}(b + ib') \wedge (c + ic') \in \Lambda_I^{2,0}$. In addition, from Lemma 4.2, we see that $d\omega_I$ is orthogonal to $\omega_I \wedge \Lambda^1$. It follows that $d\omega_I$ is primitive.

Now recall that Gray and Hervella [3], showed that the intrinsic torsion of an almost Hermitian structure (g, ω) lies in

$$\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3 + \mathcal{W}_4 = [\Lambda^{3,0}] + [U^{3,0}] + [\Lambda_0^{2,1}] + [\Lambda^{1,0}],$$

with $U^{3,0}$ irreducible: the $\mathcal{W}_1 + \mathcal{W}_2$ -part is determined by the Nijenhuis tensor; the $\mathcal{W}_3 + \mathcal{W}_4$ -part by $d\omega$. We now have from Lemma 4.2:

Proposition 4.5. *The Hermitian structure (g, ω_I, I) is of Gray-Hervella type \mathcal{W}_3 , except when $4\lambda = \mu$, when it is Kähler. Furthermore, the Kähler metric is symmetric.*

Note that the Kähler parameters do not correspond to the parameters in Theorem 4.3 that give $\xi \in [\Lambda_0^3 ES^3 H]$.

Proof. It remains to prove the last assertion. As in [8], note that $SO(7)/U(3) \cong SO(8)/U(6) \cong SO(8)/SO(2)SO(6)$, which is the quadric. The latter is isotropy irreducible and carries a unique $SO(8)$ -invariant metric with fixed volume, which is Hermitian symmetric so Kähler. However, we have seen that there is a unique Kähler metric with the same volume invariant under the smaller group $SO(7)$, so these Kähler metrics must agree. \square

Remark 4.6. Each $SO(7)$ -invariant metric g on M is given by (4.1) and so is a Riemannian submersion over $\mathbb{CP}(3)$ with fibre S^6 . The standard theory of the canonical variation [2] tell us that precisely two of these metrics are Einstein. One is the symmetric case $4\lambda = \mu$. The other is when $8\lambda = 3\mu$, as verified by Musso [6] in slightly different notation. Again these particular parameters are not those for which ξ is special.

Remark 4.7. It can be shown that the local almost Hermitian structures (g, ω_J, J) and (g, ω_K, K) above are each of strict Gray-Hervella type $\mathcal{W}_1 + \mathcal{W}_3$ at the base point, unless $4\lambda = 3\mu$, when they have type \mathcal{W}_1 . In particular, the Nijenhuis tensors N_J and N_K are skew-symmetric at the base point and equal to $\frac{1}{6}(4\lambda + \mu)(3\Phi \mp \mu^2\Psi)$ at $\text{Id } U(3)$. In [4] we showed how N_I is determined by $Jd\omega_J - Kd\omega_K$. In this case, we have the interesting situation that this latter tensor is non-zero, even though N_I vanishes. Using [1], one can prove that the obstruction to quaternionic integrability is proportional to $N_I + N_J + N_K = (4\lambda + \mu)\Phi$, confirming that this is non-zero and the results of Corollary 4.4.

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